Прості замкнені геодезичні на правильних тетраедрах у просторах постійної кривини

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Presentation Overview

Introduction

A geodesic is a locally shortest curve $\gamma : [0, 1] \to M$.
The geodesic is closed if $\gamma(0) = \gamma(1)$ and $\gamma'(0) = \gamma'(1)$.
The geodesic is called simple if it has no points of self intersection, i.e. the map $\gamma : (0, 1) \to M$ is injective.

A closed geodesics on a simply connected smooth closed two-dimensional surface appear as an integrable limit case of the planar restricted three-body problem.

In 1905 century Poincaré stated a conjecture on the existence of at least three simple closed geodesic on a smooth closed convex two-dimensional surface in Euclidean space.

This conjecture was proved by Lusternik and Schnirelman(1929), Ballman(1978), Taimanov(1992).

Since that it was created methods to find closed geodesics on regular surfaces of positive or negative curvature.

Introduction

On a closed surface of negative curvature any closed curve, that is not homotopic to zero, could be deformed to the closed curve of minimal length within its free homotopy group. This minimal curve is unique and it is a closed geodesic (Hadamard, 1898).

Let M^2 be complete closed two-dimensional Riemannian manifold of constant negative curvature.

On M^2 the number of closed geodesics of length at most L has the order of growth e^L/L as $L \to \infty$ (Huber, 1959).

On M^2 of genus g with n cusps the number of simple closed geodesics of length at most L has the polynomial order of growth $L^{6g-6+2n}$ as $L \to \infty$ (Rivin, 2001; Mirzakhani, 2008).

Introduction

Theorems about geodesics on convex two-dimensional surfaces play an important role in geometry "in the large" of convex surfaces in spaces of constant curvature.

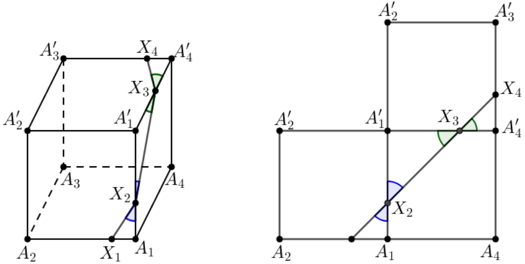
On a closed convex surface of the Gaussian curvature $\leq k$, $k > 0$, each geodesic of length $< \pi/\sqrt{k}$ realized the shortest path between its endpoints (Pogorelov, 1946).

On C^2-regular closed surface of curvature $\geq k > 0$ the length of a simple closed geodesic is $\leq 2\pi/\sqrt{k}$ (Toponogov, 1963).

Introduction

On a convex polyhedron a geodesic has following properties:

1) it consists of line segments on faces of the polyhedron;
2) it forms equal angles with an edge on the adjacent faces;
3) the geodesic cannot pass through a vertex of the convex polyhedron.



A necessary condition for the existence of a simple closed geodesic on a convex polyhedron in $\mathbb{E}^3$ follows from generalization of the Gauss-Bonnet theorem.

Introduction

D. Fuchs and E. Fuchs (2007, 2009) supplemented and systematized the results on closed geodesics on regular polyhedra in $\mathbb{E}^3$.

K. Lawson, J. Parish and others (2013) obtain a complete classification of simple closed geodesics on the eight convex polyhedra (deltahedra) whose faces are all equilateral triangles.

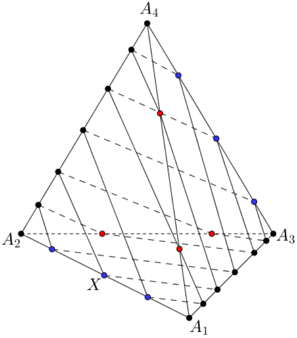
Protasov(2007) obtained a condition for the existence of simple closed geodesics on arbitrary tetrahedron in Euclidean space and evaluated from above the number of these geodesics in terms of the difference from π the sum of the angles at a vertex of the tetrahedron.

In particular, it is proved that a simplex has infinitely many different simple closed geodesics if and only if all the faces are equal triangles.

A. Akopyan and A. Petrunin (2018) showed that if closed convex surface M in $\mathbb{E}^3$ contains arbitrarily long simple closed geodesic, then M is an tetrahedron whose faces are equal triangles.

Definition

A simple closed geodesic on a tetrahedron has the type (p, q) if it has
p points on each of two opposite edges of the tetrahedron,
q points on each of other two opposite edges,
and there are $(p + q)$ points on each edges of the third pair of opposite one.



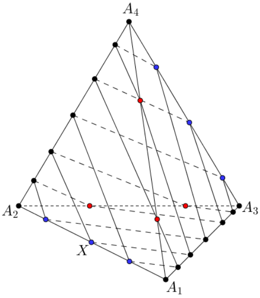
The pair of coprime integers (p, q) determines the order of intersections of the geodesic with edges of the tetrahedron, up to isometries of the tetrahedron.

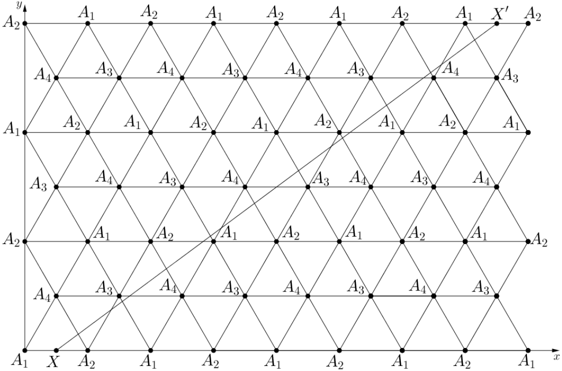
This definition doesn't depend on the curvature of the ambient space of the tetrahedron.

Outline

In Euclidean space

For any coprime integers (p, q) there exist infinitely many closed geodesics. They are parallel on the development.



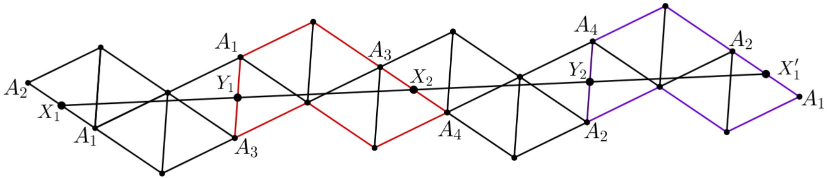


All closed geodesic on a regular tetrahedron are non-self-intersecting (simple).

Our results

Theorem 1

On a regular tetrahedron in Euclidean space, every class of simple closed geodesics γ of type (p, q) contains a simple close geodesic passing through the midpoints of two pairs of opposite edges of the tetrahedron.



Corollary 1.1

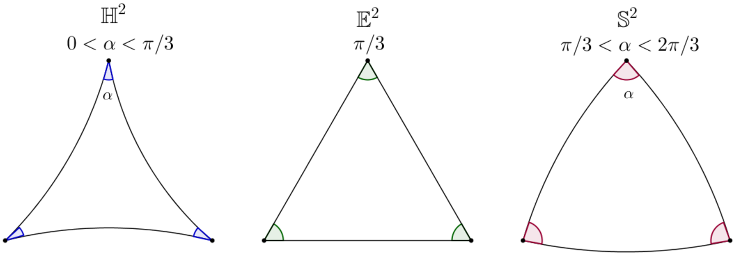
The development of the tetrahedron obtained by unrolling along γ consists of four equal polygons. Any two adjacent polygons can be transformed into each other by rotating them through an angle π around the midpoint of their common edge.

Our results

In $\mathbb{H}^3$ or in $\mathbb{S}^3$ he Gaussian curvature of the faces of a tetrahedron is -1 or 1 respectively.

The curvature of a tetrahedron is concentrated in its vertices and in its faces.

The intrinsic geometry of such tetrahedra depends on the planar angle α.



In general it is impossible to make a triangular tiling of the hyperbolic or spherical plane by regular triangles.

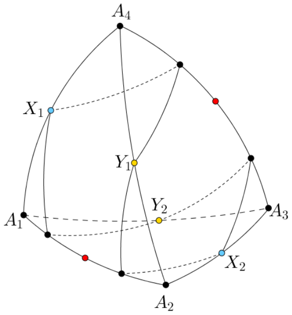
Outline

In spherical space

If $\alpha = 2\pi/3$ then the tetrahedron is a totally geodesic 2-dimensional sphere. Thus it has infinitely many simple closed geodesics.

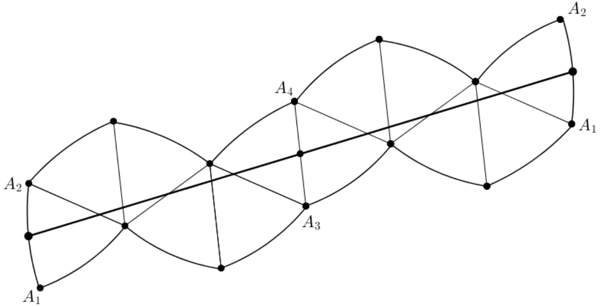
In what follow we assume $\pi/3 < \alpha < 2\pi/3$. A central projection of $S^3 \subset \mathbb{E}^4$ to the tangent space maps the regular tetrahedron from $\mathbb{S}^3$ to the regular tetrahedron in Euclidean space.

A simple closed geodesic on a regular tetrahedron in $\mathbb{S}^3$ is also characterized by a pair of coprime integers (p, q) and has the same combinatorial structure as a closed geodesic on a regular tetrahedron in Euclidean space.



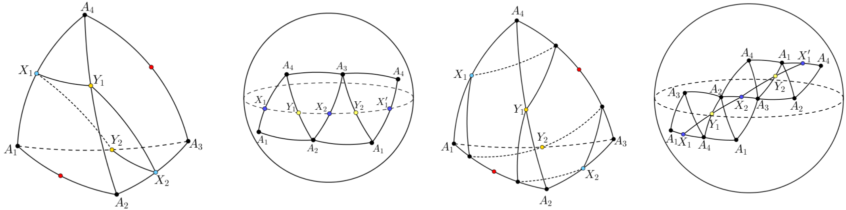
In spherical space

The development of the tetrahedron obtained by unrolling along γ consists of four equal polygons.
Any two adjacent polygons can be transformed into each other by rotating them through an angle π around the midpoint of their common edge.
On a regular tetrahedron in $\mathbb{S}^3$ a simple closed geodesic intersects midpoints of two pairs of opposite edges.



Lemma 1

1) On a regular tetrahedron with the planar angle $\alpha \in (\pi/3, 2\pi/3)$ in $\mathbb{S}^3$ there exist three different simple closed geodesics of type $(0,1)$. They coincide under isometries of the tetrahedron.
2) Geodesics of type $(0,1)$ exhaust all simple closed geodesics on a regular tetrahedron with the planar angle $\alpha \in [\pi/2, 2\pi/3)$ in $\mathbb{S}^3$.
3) On a regular tetrahedron with the planar angle $\alpha \in (\pi/3, \pi/2)$ in $\mathbb{S}^3$ there exist three different simple closed geodesics of type $(1,1)$.



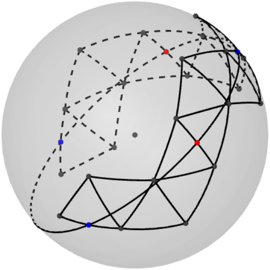
In spherical space

Lemma 2

The length of a simple closed geodesic on a regular tetrahedron in $\mathbb{S}^3$ is less than 2π.

It was proved analysing the construction of a simple closed geodesic on a regular tetrahedron.

However, it can be considered as the particular case of the result proved by A. Borisenko (2020) about the generalization of V. Toponogov theorem (1963) to the case of two-dimensional Alexandrov space.



In spherical space

Theorem 2

On a regular tetrahedron with the planar angle α in $\mathbb{S}^3$ such that

$$\alpha > 2 \arcsin \sqrt{\frac{p^2 + pq + q^2}{4(p^2 + pq + q^2) - \pi^2}},$$

where (p, q) is a pair of coprime integers, there is no simple closed geodesic of type (p, q).

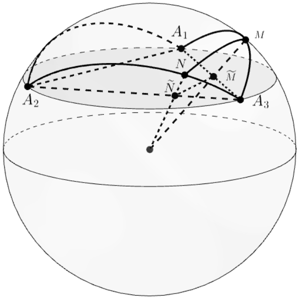
On a regular tetrahedron in spherical space with the planar angle $\pi/3 < \alpha < 2\pi/3$ there exists the finite number of simple closed geodesic.

In spherical space

Idea of the proof of Theorem 2:

We constructed a special geodesic map of a regular tetrahedron in $\mathbb{S}^3$ to a regular tetrahedron in $\mathbb{E}^3$.

It gave us an estimation from below for the length of a simple closed geodesic of type (p, q) on the regular tetrahedron in spherical space.



$$L_{p,q} > 2\sqrt{p^2+pq+q^2}\,\frac{\sqrt{4\sin^2(\alpha/2)-1}}{\sin(\alpha/2)} \geq 2\pi.$$

In spherical space

Theorem 3

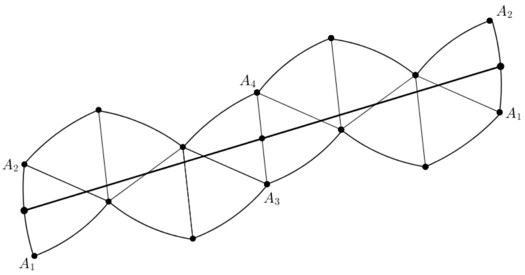
For any pair of coprime integers (p, q) there exists $\varepsilon_0 \in (0, \pi/6)$ such that on a regular tetrahedron in spherical space with the planar angle $\alpha \leq \pi/3 + \varepsilon_0$ there exists simple closed geodesic γ of type (p, q).
The geodesic γ is unique, up to the rigid motion of the tetrahedron, and γ passes through the midpoints of two pairs of opposite edges of the tetrahedron.

In spherical space

Idea of the proof of Theorem 3:

Construct a (p, q) development $D(\alpha)$ the regular tetrahedron onto the unit sphere S^2.

For fixed (p, q) consider a one-parameter family of closed polygons $D(\alpha)$, where $\alpha \in (\pi/3, 2\pi/3)$.

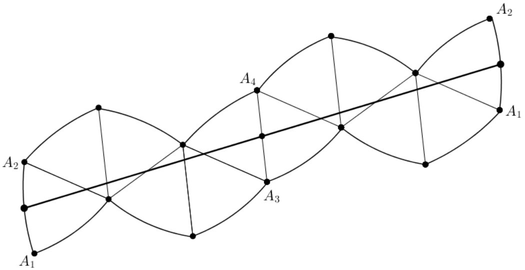


A development has property of symmetry.

In spherical space

Idea of the proof:

A development has property of symmetry.

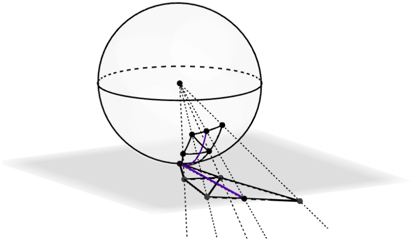


Consider the broken line $\gamma(\alpha)$ on $D(\alpha)$ that connect the points of symmetry X_1, X_2, Y_1, Y_2, X_1'.

If $\gamma(\alpha)$ lies inside $D(\alpha)$, then $\gamma(\alpha)$ corresponds to the simple closed geodesic on regular tetrahedron.

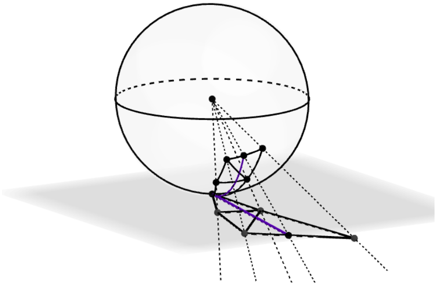
In spherical space

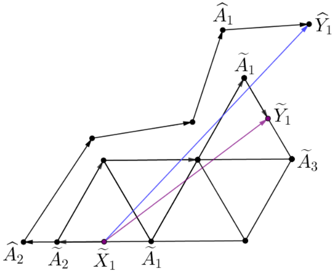
Idea of the proof:
If $\alpha \to \pi/3$, then the length of the edge $a \to 0$.
Consider a homothety of this unite sphere S^3 with the origin at the center of the sphere and a coefficient $\lambda = 1/a$.
In this case the curvature of the faces of the tetrahedron is a^2.



In spherical space

Idea of the proof:
Apply a central projection of S^2 to $\mathbb{E}^2$.
h is a min distance from the vertices of the development to the simple closed geodesic inside it.
If $\alpha = \pi/3 + \varepsilon$, then $h_S > h_E - c(p,q)\varepsilon$.





Theorem 5

For any pair of coprime integers (p, q) there exists $\varepsilon_0 \in (0, \pi/6)$ such that on a regular tetrahedron in $\mathbb{S}^3$ with the planar angle $\alpha \leq \pi/3 + \varepsilon_0$ there exists simple closed geodesic γ of type (p, q).

The geodesic γ is unique, up to the rigid motion of the tetrahedron, and γ passes through the midpoints of two pairs of opposite edges of the tetrahedron.

In spherical space

Consider rectifiable curves $\gamma(\alpha)$ in $D(\alpha)$ that connect the points of symmetry X_1, X_2, Y_1, Y_2, X_1' inside $D(\alpha)$.

If X_1X_1' is a straight line that lies inside the development $D(\alpha)$, then $\gamma(\alpha)$ corresponds to the simple closed geodesic on regular tetrahedron $T(\alpha)$.

Then the length of $\gamma(\alpha)$ is less then 2π.

The infimum $L_{p,q}(\alpha)$ of the lengths of the curves $\gamma(\alpha)$ is referred to as the length of the abstract shortest curve in the development.

Theorem 6 (Borisenko 2022)

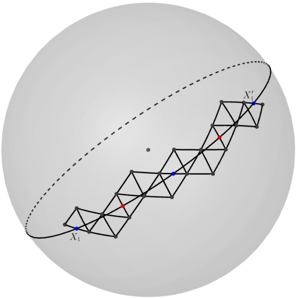
On a regular tetrahedron in spherical space of curvature one there exist a simple closed geodesic of type (p,q) if and only if the length of the abstract shortest curve on the development $L_{p,q}(\alpha) < 2\pi$.

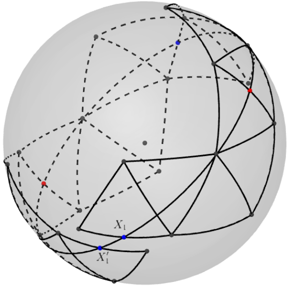
In spherical space

Let us increase α starting from $\pi/3 + \varepsilon_0$

$D(\alpha)$ is considered as an abstract polygon homeomorphic to a disc, with intrinsic metric, since each interior point of this polygon has a neighbourhood isometric to the interior of a disc on the unit sphere $\mathbb{S}^2$.

This polygon is locally isometrically immersed in the sphere $\mathbb{S}^2$





In spherical space

From necessary condition of Theorem 6 it follows:

If the value α satisfies the inequality

$$\alpha > 2 \arcsin \sqrt{\frac{p^2 + pq + q^2}{4(p^2 + pq + q^2) - \pi^2}},$$

then in spherical space on a regular tetrahedron with faces angle α there is no simple closed geodesic of type (p, q).

From sufficient condition of Theorem 6 it follows:

If the length a of the edges of a regular tetrahedron in spherical space of curvature one satisfies the inequality

$$a < 2 \arcsin \frac{\pi}{\sqrt{p^2 + pq + q^2} + \sqrt{(p^2 + pq + q^2) + 2\pi^2}},$$

then there exists a unique simple closed geodesic of type (p, q) on such tetrahedron, up to the rigid motion.

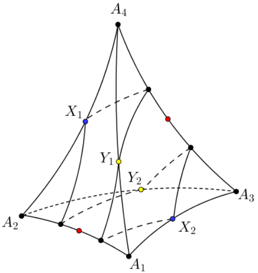
Outline

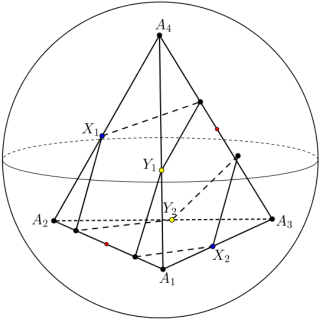
In hyperbolic space

The planar angle satisfies $0 < \alpha < \pi/3$.

In the Cayley-Klein model of hyperbolic space the regular tetrahedron in $\mathbb{H}^3$ is represented by a regular tetrahedron in $\mathbb{E}^3$.

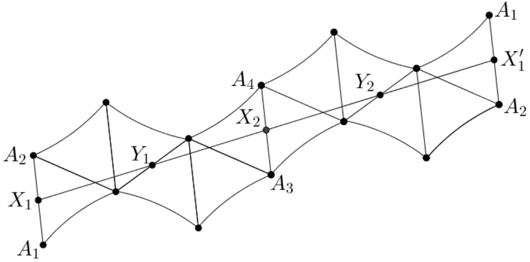
A simple closed geodesic γ on a regular tetrahedron in $\mathbb{H}^3$ is also characterized by a pair of coprime integers (p, q).





In hyperbolic space

The development of the tetrahedron along γ has the property of symmetry.

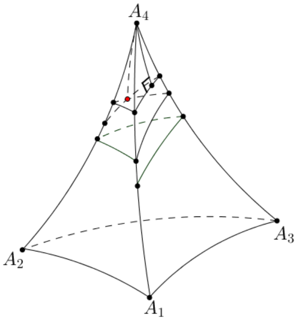


On a regular tetrahedron in $\mathbb{H}^3$ a simple closed geodesic intersects midpoints of two pairs of opposite edges.

In hyperbolic space

Lemma 3

If a geodesic on a regular tetrahedron in $\mathbb{H}^3$ intersects three edges meeting at a common vertex consecutively, and intersects one of these edges twice, then this geodesic has a point of self-intersection.



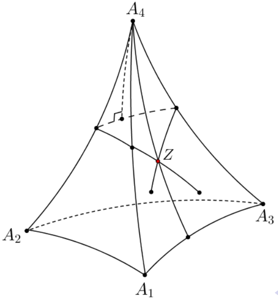
In hyperbolic space

Lemma 4

Let d be the minimum distance from the vertices of a regular tetrahedron in $\mathbb{H}^3$ to a simple closed geodesic on the tetrahedron. Then

$$d > \frac{1}{2} \ln \left(\frac{\sqrt{2\pi^3} + (\pi - 3\alpha)^{\frac{3}{2}}}{\sqrt{2\pi^3} - (\pi - 3\alpha)^{\frac{3}{2}}} \right),$$

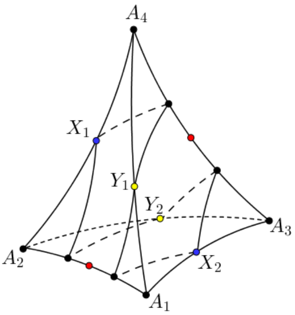
where α is the planar angle of a face of the tetrahedron.



In hyperbolic space

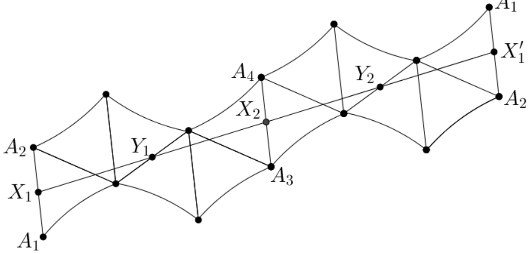
Theorem 5

On a regular tetrahedron in $\mathbb{H}^3$ for each ordered pair of coprime integers (p, q), there exists unique, up to the rigid motion of the tetrahedron, simple closed geodesic of type (p, q).
Geodesics of type (p, q) exhaust all simple closed geodesics on a regular tetrahedron in hyperbolic space.



In hyperbolic space

Idea of the proof of Theorem 5:
A development has property of symmetry.
If $\alpha \leq \pi/4$, then this development is the convex polygon and contains a simple closed geodesic γ of type (p, q).
If $\alpha > \pi/4$, then the development still contains γ inside.
Since $d(\alpha) > 0$ for $\alpha \in (0, \pi/3)$, we can increase α to $\pi/3$ without losing simple closed geodesic of type (p, q) inside.



In hyperbolic space

Theorem 6

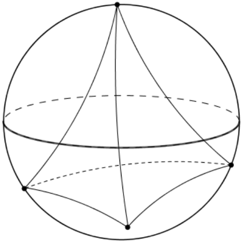
Let $N(L, \alpha)$ be the number of simple closed geodesics of length $\leq L$ on a regular tetrahedron with the planar angle α in $\mathbb{H}^3$. Then

$$N(L, \alpha) = c(\alpha)L^2 + O(L \ln L),$$

where $O(L \ln L) \leq CL \ln L$ as $L \to +\infty$, and $\lim\limits_{\alpha \to 0} c(\alpha) = c_0 > 0$.

When $\alpha \to 0$, the vertices of the tetrahedron goes to infinity. Then the limiting tetrahedron is a noncompact surface homeomorphic to a sphere with four cusps, with a complete regular Riemannian metric of constant negative curvature.

In work of Rivin(2001) it was shown that the number of simple closed geodesics on this surface has order of growth L^2.



Outline

Submanifolds of revolution

Definition

A multidimensional Riemannian metric on a manifold F^l is called a metric of revolution if there exists a regular coordinate system such that this Riemannian metric has the form

$$ds^2 = \left(du^1\right)^2 + \varphi^2(u^1)d\sigma^2,$$

where $\varphi(u^1) > 0$ is a regular function, $d\sigma^2$ is a Riemannian metric of constant sectional curvature.

Submanifolds of revolution

In E^3 a two-dimensional surface of revolution F^2 of constant Gaussian curvature admits a standard coordinate system such that the metric of F^2 is the metric of revolution.

On the other hand from the fact that the induced metric on $F^2 \subset E^3$ is a metric of revolution it does not follow that F^2 is a surface of revolution.

There is a locally isometric embedding F^2 into E^3 such that the geodesic line $u^2 = 0$ is mapped onto a space curve with torsion not equal to zero at any point.

Borisenko(2018) answered this question in a case when the ambient space is Euclidean.

The submanifold of low codimension in Euclidean space with induced metric of revolution is a submanifold of revolution if the coordinate geodesic lines are the lines of curvature.

Submanifolds of revolution

Let E_1^n be a pseudo euclidean space of signature $(1, n)$ with the scalar product

$$\langle X, Y\rangle = -x^0y^0 + x^1y^1 + \cdots + x^ny^n.$$

Consider hyperbolic space as a sheet of hyperboloid with induced metric

$$H^n = \{X(x^0, x^1, \ldots, x^n) | \langle X, X\rangle = -1, x_0 > 0\}.$$

The curve $\gamma(u^1) = \left(\chi(u^1), \psi(u^1), \varphi(u^1), 0, \ldots, 0\right)$ lies on $H^2 \subset E_1^2$ and u^1 is the arc-length parameter of γ.
Rotate $\gamma(u^1)$ along the submanifold $F^{l-1} \subset S^{l+p-2}$ in E_1^{l+p}.
The radius vector of F^{l-1} is

$$\rho(u^2, \ldots, u^l) = \left(0, 0, \rho^1(u^2, \ldots, u^l), \ldots, \rho^{l+p-1}(u^2, \ldots, u^l)\right).$$

The submanifold F^{l-1} has the intrinsic Riemannian metric $d\sigma^2$ of constant sectional curvature.
Thus we get a submanifold of revolution F^l.

Submanifolds of revolution

Definition

A submanifold F^l in hyperbolic space $H^{l+p} \subset E_1^{l+p}$ is called a submanifold of revolution, if the radius vector of F^l equals

$$r(u^1, \dots, u^l) = \begin{cases} x^0 = \chi(u^1); \\ x^1 = \psi(u^1); \\ x^2 = \varphi(u^1)\rho^1(u^2, \dots, u^l); \\ \dots; \\ x^{l+p} = \varphi(u^1)\rho^{l+p-1}(u^2, \dots, u^l); \end{cases}$$

where

$$-\chi^2 + \psi^2 + \varphi^2 = -1;$$

$$\left(\rho^1\right)^2 + \left(\rho^2\right)^2 + \dots + \left(\rho^{l+p-1}\right)^2 = 1.$$

and $d\sigma^2 = \left(d\rho^1\right)^2 + \dots + \left(d\rho^{l+p-1}\right)^2$ is a Riemannian metric of constant sectional curvature.

Submanifolds of revolution

Definition

A line $\gamma \subset F^l \subset E_1^{l+p}$ is called a line of curvature of a submanifold F^l if for any normal n from the normal space NF^l the tangent vector γ' is a principal direction of the second fundamental form with respect to the normal n.

Negative extrinsic sectional curvature

Theorem 7

Let F^l be a C^3-regular submanifold in $H^{2l-1} \subset E_1^{2l-1}$ with the induced metric of revolution of negative extrinsic sectional curvature

$$ds^2 = \left(du^1\right)^2 + \varphi^2(u^1)d\sigma^2,$$

where where $\varphi(u^1)$ is a regular positive function and $d\sigma^2$ is a Riemannian metric of constant sectional curvature. If the coordinate lines u^1 are lines of curvature of the submanifold F^l, then F^l is a submanifold of revolution.

Zero extrinsic sectional curvature

Theorem 8

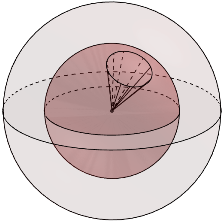
Suppose F^l is a regular hypersurface in hyperbolic space H^{l+1} with the induced metric of revolution of zero extrinsic sectional curvature
Let the coordinate lines u^1 be the lines of curvature of the submanifold F^l.

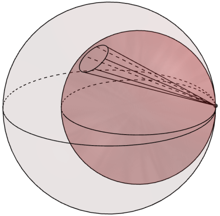
1. If $d\sigma^2$ is a metric of constant sectional curvature -1, then F^l is a cylinder with one-dimensional generator over a local isometric immersion of a domain of hyperbolic space H^{l-1} into H^l.
2. If $d\sigma^2$ is a metric of constant sectional curvature 1, then F^l is a cone with one-dimensional generator over a local isometric immersion of a domain of the unit sphere S^{l-1} into the unit sphere $S^l \subset H^{l+1}$.
3. If $d\sigma^2$ is a metric of constant sectional curvature 0, then F^l is an asymptotic cone with one-dimensional generator over a local isometric immersion of a domain of Euclidean space E^{l-1} into the horosphere $E^l \subset H^{l+1}$.

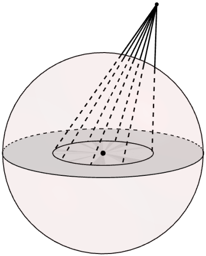
Zero extrinsic sectional curvature

Let L^l be a hypersurface of constant curvature in H^{l+1}.
Let F^{l-1} be a submanifold of L^l. Through every point of F^{l-1} construct the geodesics γ tangent to the normal of L^l in H^{l+1}. We get the surface F^l with one-dimensional generator over the submanifold F^{l-1} in H^{l+1}.
Consider H^{l+1} in a Cayley-Klein model inside the unit ball. Then

1) If all geodesics γ intersect in the fixed point inside the ball, then F^l is a cone.
2) If all geodesics γ intersect in the fixed point on the absolute of the model, then F^l is called an asymptotic cone.
3) If all geodesics γ do not intersect each other either inside the ball or on the absolute, then F^l is a cylinder with one-dimensional generator.







Positive extrinsic sectional curvature

Theorem 9

Suppose F^l is a regular hypersurface in hyperbolic space $H^{l+1} \subset E_1^{l+1}$ with the induced metric of revolution of positive extrinsic sectional curvature.

1. If $l > 2$, then F^l is a hypersurface of revolution.
2. If $l = 2$ and the coordinate lines u^1 are lines of curvature, then F^2 is a hypersurface of revolution in H^3.

To prove this statement we used a Pogorelov transformation of a hyperbolic space into Euclidean one.

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Thank you for attention!